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Variations of the Jacobian "Constants" in the Restricted Four-Body Problem

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Abstract

This paper is a presentation of some unpublished results, derived in the early 1970s, showing that the zero-sphere-of-influence patched conic method, used for many years in preliminary lunar and interplanetary mission design, satisfies the Jacobian integral of the restricted three-body problem, to within terms of the order of the mass ratio of the two primaries, and terms of the order of the normalized passage distance. The integral formalism is extended to discussions of the restricted four-body problem, in which the constants of the two superimposed three-body problems of the Earth-moon/Sun-Earth systems are treated as variables whose limits can be roughly estimated to predict extreme values of energy and angular momentum achievable through multiple excursions to the separate regions of three-body dominance.

Introduction

The original purpose of this paper was to publish some early results which, to date, have appeared only as a memo written while the author was with Douglas Aircraft Company and, shortly thereafter, as a white paper written while the author was with Analytical Mechanics Associates, Inc. Since that time, the author has referred repeatedly to this development with the familiar phrase "It can be shown that ...." It seemed appropriate that the development should be published so that the familiar phrase could be followed by a published reference. Later, it became apparent that some practical applications might be suggested.

Practically, the development allows one to have confidence in the zero-sphere-of-influence or "point-to-point" patched conic method insofar as it is used to predict the moon-relative energy during lunar closest approach. This formulation, which exposes the terms of the Jacobian integral not always included in discussions of practical applications, might be useful in obtaining estimates of invariant quantities of trajectories near the transition region from the Earth-moon system to the Sun-Earth system. The following sections contain the development of the zero-patched conic from Jacobi's integral, discussions of the application of the approximations to the Sun-Earth transition region, and some suggestions as to how the famous integral of Jacobi can be used to help identify regions of useful transfer dynamics in the Sun/Earth-moon system.

The Zero-Patched Conic and Jacobi's Integral

In Brouwer and Clemence[1], it is shown that the Jacobian integral in the circular restricted three-body problem can be expressed relative to a non-rotating, primary-centered frame as

$$E - n' \, h \cos i = \mu' \left\{ \frac{1}{\rho} - \frac{\boldsymbol{r} \bullet \boldsymbol{r}'}{a'^{3}} \right\} - C \ , \tag{1}$$

where the symbols have the following meanings:

- E Energy (per unit mass) relative to the central primary
- n' mean motion of the perturbing body
- h angular momentum (per unit mass) relative to the central primary
- i inclination of the osculating Keplerian orbit relative to the plane of the perturbing planet's motion
- μ' gravitational constant of the perturbing body
- ρ distance of the (massless) particle from the perturbing body
- $\boldsymbol{r}$ position vector of the particle in the central primary frame
- $\boldsymbol{r}'$ position vector of the perturbing body in the central primary frame
- a' semi-major axis of the perturbing body's orbit

  Note: If μ is the gravitational constant of the central primary,

  $n'^2 a'^3 = \mu + \mu'$
- C Jacobi's constant, a quantity which has the units of energy (per unit mass) and remains constant for all time in the circular restricted 3-body problem

Henceforth, in this section, the central primary will be called the Earth and the perturbing body will be called the moon. The third body (whose mass is assumed not to affect the circular orbits of the two primaries about their center of mass) will be referred to as the particle or the vehicle.

At some time, t_1, let the particle have initial conditions denoted by a subscript 1. At t_1, then,

$$E_1 - n' \, h_1 \cos i_1 = \mu' \left\{ \frac{1}{\rho_1} - \frac{r_1 \cos S_1}{a'^{2}} \right\} - C \ , \tag{2}$$

where S_1 is the angle between the vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}'_1$. Since t_1 is the initial point, we can calculate C exactly in terms of the initial conditions.

Now let the particle make a close approach to the moon by passing within a distance ε of that body.

At closest approach time, t_2, we have

$$E_2 - n' \; h_2 \cos i_2 = \mu' \left\{ \frac{1}{\rho_2} - \frac{r_2 \cos S_2}{a'^2} \right\} - C \; . \tag{3}$$

We commit a very small error (for small ε) if we replace the second term in brackets with 1/a′ to obtain

$$E_2 - n' \; h_2 \cos i_2 = \frac{\mu'}{\varepsilon} - \frac{\mu'}{a'} - C \pm \frac{\mu' \varepsilon}{a'^2} \; , \tag{4}$$

where the last term represents the maximum possible error. Errors of order ε/a′ are incurred if the left-hand side of (4) is rewritten as if there were no difference between a′ and r_2 and by the (equivalent) assumption that S2 ≈ ε/a′. We then obtain, neglecting terms of order ε/a′,

$$\frac{V_2^2}{2} - \frac{\mu}{a'} - n' \; d \; V_2 \cos \gamma_2 \cos i_2 = \frac{\mu'}{\varepsilon} - \frac{\mu'}{a'} - C \; , \tag{5}$$

where V_2 and $\gamma_2$are the magnitude and path angle of the geocentric velocity at lunar closest approach.

The term $V_2^2/2$ can be rewritten in terms of a moon-relative speed, V_2^*, as

$$\frac{V_2^2}{2} = \frac{V_2^{*2}}{2} - \frac{V_M^2}{2} + V_2 V_M \cos \gamma_2 \cos i_2 \; , \tag{6}$$

where V_M is the (circular) geocentric speed of the moon. Using (6) in (5), and noticing that $n' = V_M/a'$, we get

$$\frac{V_2^{*2}}{2} - \frac{\mu'}{\varepsilon} = \frac{V_M^2}{2} + \frac{\mu - \mu'}{a'} - C \; . \tag{7}$$

But the left-hand side of (7) is just the moon-relative energy of the particle at closest approach,

$$E_2^* = \frac{V_M^2}{2} + \frac{\mu - \mu'}{a'} - C \; ,$$

or, in terms of the moon-relative hyperbolic excess speed,

$$V_\infty^{*2} = V_M^2 + \frac{2\left[\mu - \mu'\right]}{a'} - 2C \; . \tag{8}$$

Now, we return to the calculation of the Jacobian constant from initial conditions. Rewriting (2) in terms of the initial elements gives

$$C = n' \, h_1 \cos i_1 - \left[\frac{V_1^2}{2} - \frac{\mu}{r_1} \right] + \mu' \left\{ \frac{1}{\rho_1} - \frac{r_1 \cos S_1}{a'^2} \right\} .$$

In terms of the initial Keplerian orbit with respect to the Earth, where V_{2K} represents the speed on the initial Keplerian ellipse at the time when the particle is a distance r_2 = a′ from the Earth,

$$C = n' \, d \, V_{2K} \cos \gamma_{2K} \cos i_K - \left[\frac{V_{2K}^2}{2} - \frac{\mu}{a'} \right] + \frac{\mu'}{a'} \, , \qquad (9)$$

with similar meaning for γ_{2K}, and i_K is the initial inclination of the orbit to the Earth-moon plane. In the above, we have simply replaced the initial Keplerian elements with the equivalent expressions calculated at a time when the particle is a distance a′ from the Earth. The perturbation term in curly brackets has been replaced with μ'/a′, the limiting value for the situation where the initial distance of the particle from the Earth is small compared with the distance to the disturbing body. Substituting (9) into (8) yields

$$V_\infty^{*2} = V_M^2 + V_{2K}^2 - 2 \, V_M \, V_{2K} \cos \gamma_{2K} \cos i_K - \frac{4\mu'}{a'} \, . \qquad (10)$$

Except for the term $-4\mu'/a'$, equation (10) is simply the expression for the moon-relative excess speed given by the zero-sphere-of-influence patched conic method where the particle is assumed to travel to the center of the moon, the Keplerian velocity is transformed to a moon-centered (non-rotating) frame, and the result is taken to be the moon-relative hyperbolic excess velocity. Dropping the last term in (10) shows that the zero-patched conic method conserves Jacobi’s integral to within terms of the order of the normalized passage distance and the normalized mass ratio in the circular restricted 3-body problem. Except for the transformation to laboratory coordinates, this is identical with the method used by Professor Rutherford[2] in his analysis of the α-particle scattering problem near the turn of the 20th century. The congruity of the method with the Jacobian integral in the Earth- to-moon problem is the reason we are able to predict so accurately the moon-passage energy, and therefore the orbit insertion requirements, using what is apparently straightforward two-body dynamics and Lambert’s theorem. Of course, the method tells us nothing of the angular momentum during moon-passage; all we know is that it is assumed to be small, and easily controlled by midcourse maneuvers.

The Same Problem Backwards

Now. let us reverse the process of the Earth-to-moon transfer described in the previous section and apply it to the problem of solar perturbations acting on a vehicle in the Earth-moon system. Suppose we start from a position near the Earth on a trajectory that nearly escapes the Earth-moon system. Fig. 1 is a diagram of a lunar swingby to a point near the region where the solar gravitational perturbations are fairly strong. The figure shows the return to the Earth and the escape from the Earth-moon system.

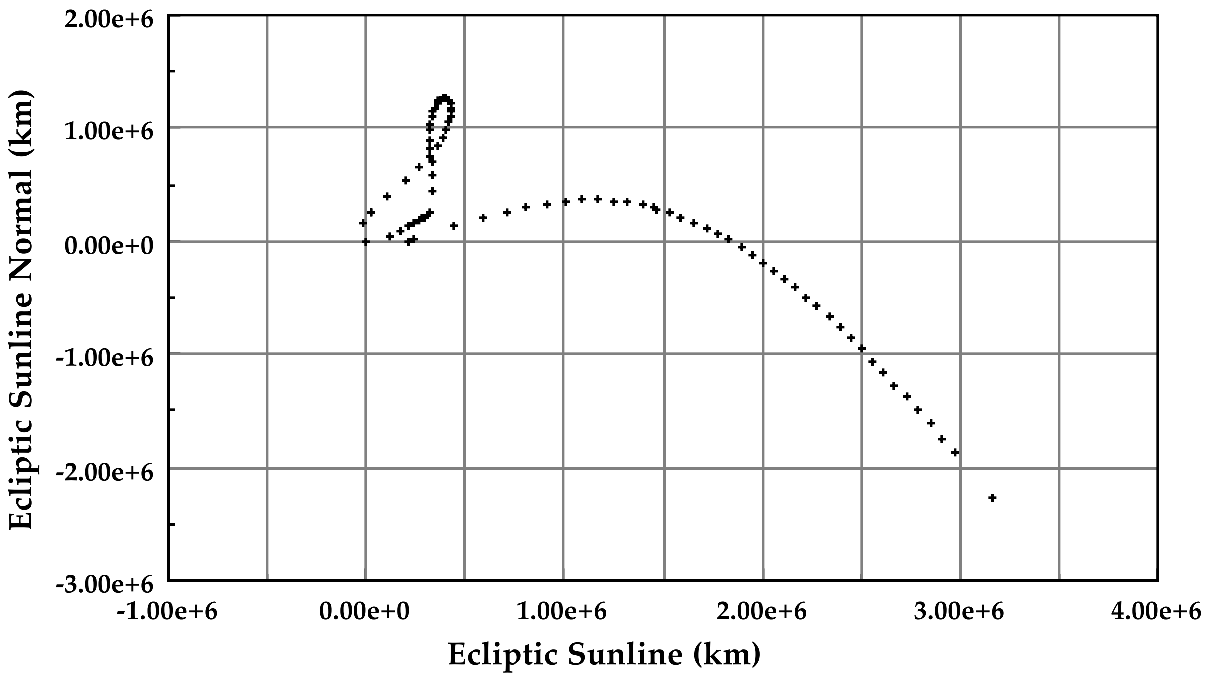

Fig. 1 Lunar Swingby to Transition Region and Escape from Earth-moon System

The time history of the Jacobian constant for the Sun/Earth-moon three-body problem is shown in Fig. 2. Notice that the perturbation term is much larger in this case than the dynamical terms representing the Earth-relative energy and axial angular momentum. In this case the quantity is

$$C_s = n_s\, h \cos i - \left[\frac{V^2}{2} - \frac{\mu}{r}\right] + \mu_s \left\{\frac{1}{\rho_s} - \frac{r \cos S_s}{a_s^2}\right\},$$

where the subscript s refers to the Sun and the other quantities are with respect to an Earth-centered, non-rotating frame with its xy plane in the ecliptic. All numerical calculations done for this study have been numerically integrated using a four-body model with Earth, moon, and Sun in the ecliptic and with circular orbits about each other. In Fig. 2, we see the sharp change in C_s due to the lunar swingby at about 6 days after which the "constant" of the Earth-moon/Sun problem is very nearly constant. Note the small "blip" at about 65 days when the vehicle makes a fairly close approach (about 130,000 km) to the moon. The message is clear in Fig. 2; only the moon can change the value of C_S, and, as we shall see, only the Sun can change the value of C for the Earth-moon-vehicle system.

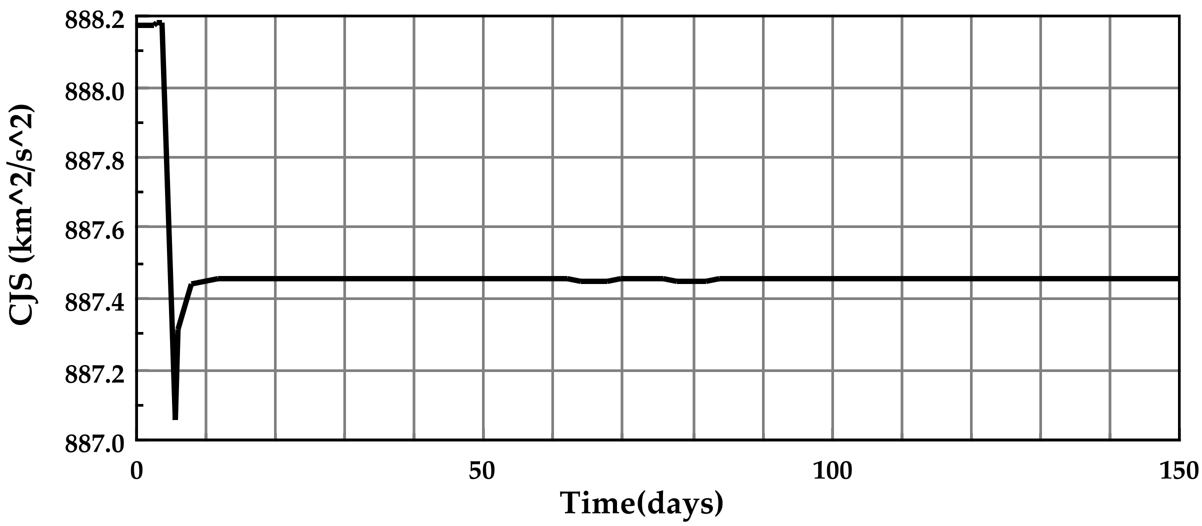

Fig. 2 Variation of Cs for the Transfer of Fig. 1

For the same trajectory, the value of C , the Jacobian constant for the Earth-moon system, as discussed in the previous section is plotted in Fig. 3. Here, the effect of the Sun causes considerable variation in the value of C. The analyst should be cautioned that, because the largest part of C_S is due to the perturbation term in curly brackets ($\approx \mu_S / a_S = 887.04\ \mathrm{km}^2/\mathrm{s}^2$) , its value is very much affected by the variation of the distance from the Sun to the Earth-moon system. In these studies, the Sun's orbit is circular; in real-world studies, one should account for the variation in distance with time.

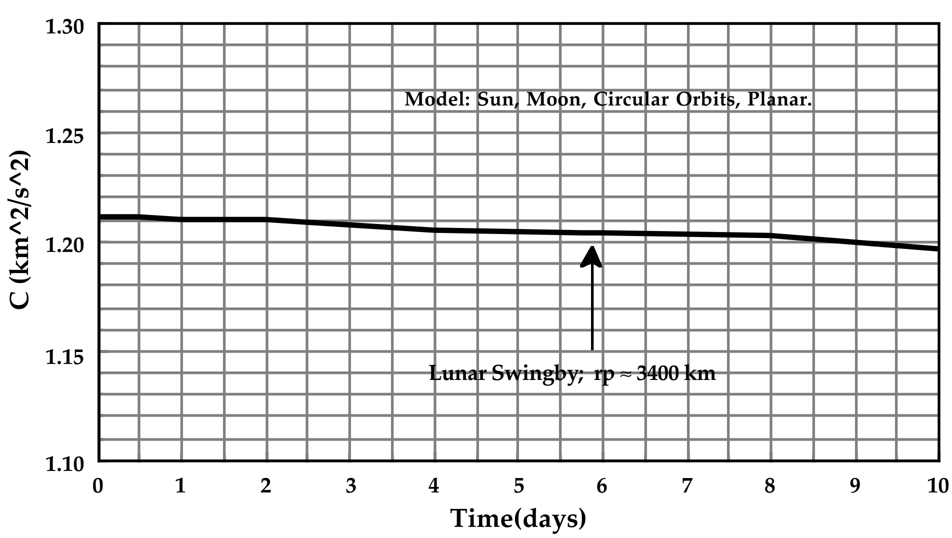

Fig. 3 Variation of Earth-moon Jacobian Constant for Transfer of Fig. 1

The longer-term variation of C is shown in Fig. 4. Here, we see the power of the solar perturbations to change the value of C for the Earth-moon system. The decrease in the value of C, for an orbit whose energy is negative, indicates that the solar perturbations are decreasing the value of the axial angular momentum, $h \cos i$. It is this kind of variation that will permit a substantial change in the value of the moon-relative excess speed at subsequent close encounters with the moon. It is the value of the excess speed, as given by (8) above, that will ultimately determine the amount of energy to be gained by a sequence of close encounters with the moon.

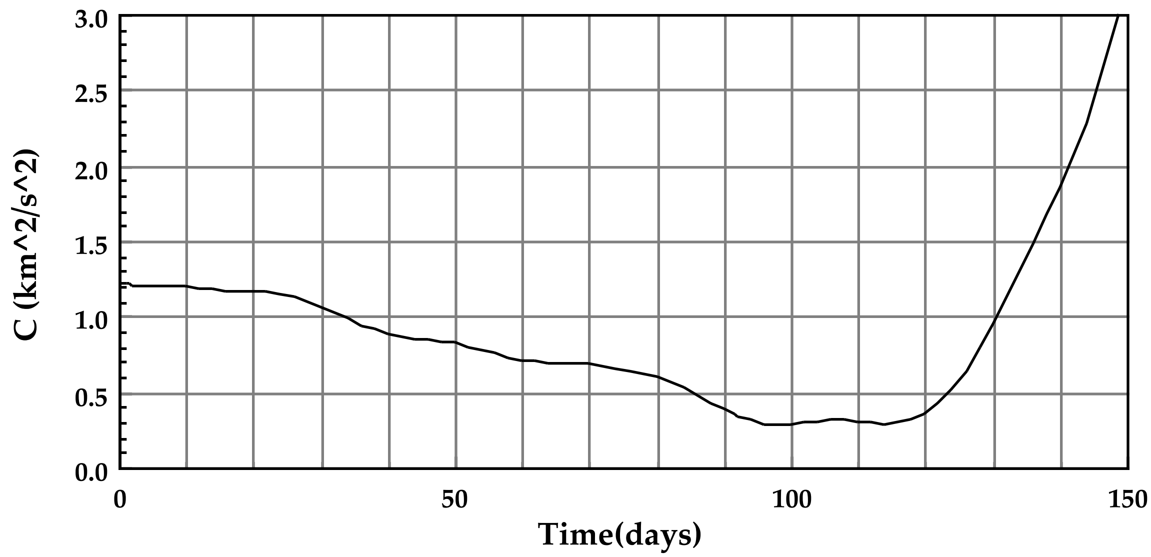

Fig. 4 Variations in *C* due to Solar Perturbations

Application to the Triple Lunar Swingby

For some time, the author has been interested in determining the maximum Earth-relative energy change available from a sequence of close lunar swingbys[3,4]. Most recently[5], a method for using three close lunar encounters was presented in which a near-minimal energy (C_3 = -2 km^2/s^2) was boosted to a C_3 of + 4.4 km^2/s^2. Since the presentation of that paper, Dunham[6] has pointed out that he and his colleagues had generated a Triple Lunar Swingby (TLS) in 1991. This trajectory was presented to the participants of a workshop at ISAS on missions to near-Earth bodies, but the results were not published. Of course, Farquhar, Dunham, and Hsu[7] had already demonstrated the two essential elements of the TLS - first, transfer to the region of strong solar perturbations, so as to change the Earth-relative angular momentum without undue (negative) change in energy, and second, the use of a retrograde double lunar swingby to boost the spacecraft well beyond escape from the Earth-moon system. In this section, we take up the application of the principles outlined above to the understanding of trajectory shaping mechanisms near the transition region of the Sun-Earth-moon system.

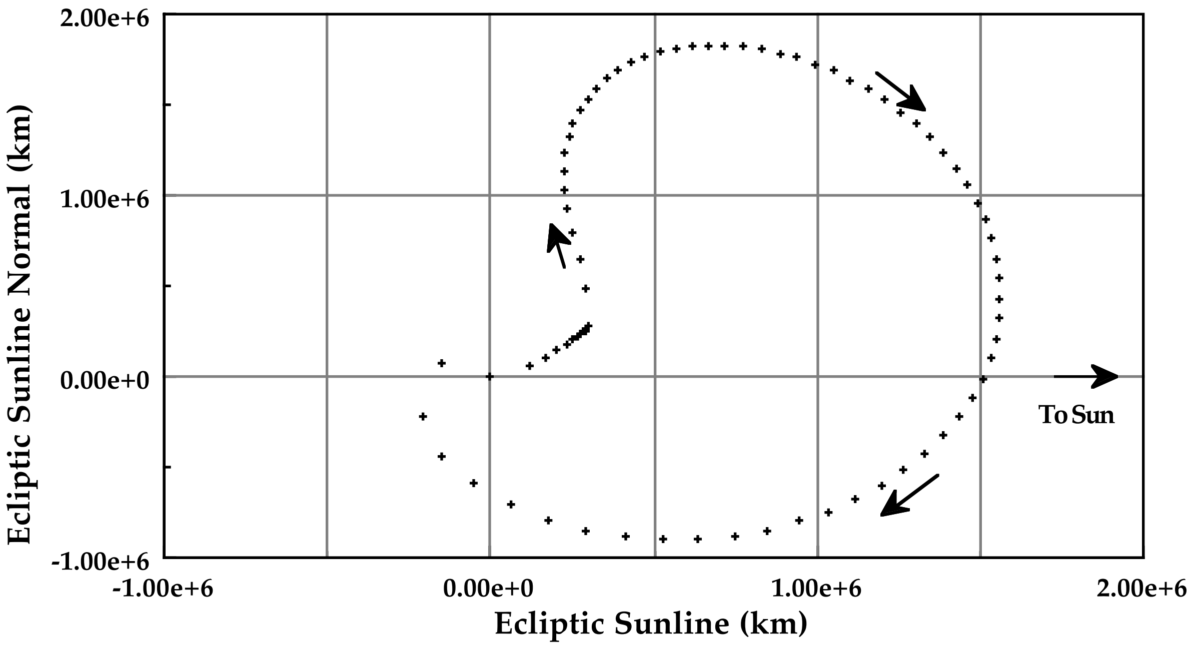

Fig. 5 Angular Momentum Reversal Phase of the Triple Lunar Swingby

Fig. 5 shows the early portion of a TLS generated in the context of the planar, circular restricted four-body problem. Notice that the trajectory (after the initial energy boosting lunar swingby, S_1) starts out nearly normal to the Earth-Sun line. The trajectory is shown at 12 hour intervals for the first four days, and at two day intervals for the remainder of the 150 day simulation. As the spacecraft approaches 1.5 to 2 million kilometers from Earth, the solar perturbations begin to reverse the angular momentum with respect to the non-rotating Earth frame (the plot frame of Fig. 5 is rotating, however) but, at this distance, the disturbing acceleration has little effect on the spacecraft's Earth-relative energy. The entire object of this phase of the TLS is to use the lunar gravity to boost a near-minimal energy Earth-to-moon transfer to the transition region of the Sun-Earth system so as to set up the conditions for a retrograde double-lunar swingby, suggested by Ross[8], and later elaborated by Bender[9]. The technical objective of the maneuver is to acquire enough retrograde angular momentum that the first of the two retrograde lunar swingbys (S_2, the second swingby of the TLS) will not cause the spacecraft to impact the Earth while still maintaining a retrograde transfer in order to ensure the third and final energy-boosting lunar encounter, S_3.

Fig. 6 shows the time histories of the Earth-relative energy and axial angular momentum (scaled by the mean motion of the moon). In the example of the previous section, we saw that the Sun could significantly affect the Jacobian constant of the spacecraft in the Earth-moon system. Now, by "shooting" a little harder (using a closer 1st lunar swingby of about 1950 km radius) and by directing the transfer a little more along the y axis, we can cause the spacecraft to spend more time in the 1st quadrant of the sun-pointing reference system and thereby gain substantial decrease in the Earth-relative angular momentum.

The time history of the distance from Earth to spacecraft is shown in Fig. 6b aligned with Fig. 6a to show the relationships between distance and the "constants" of the two, superimposed three-body problems.

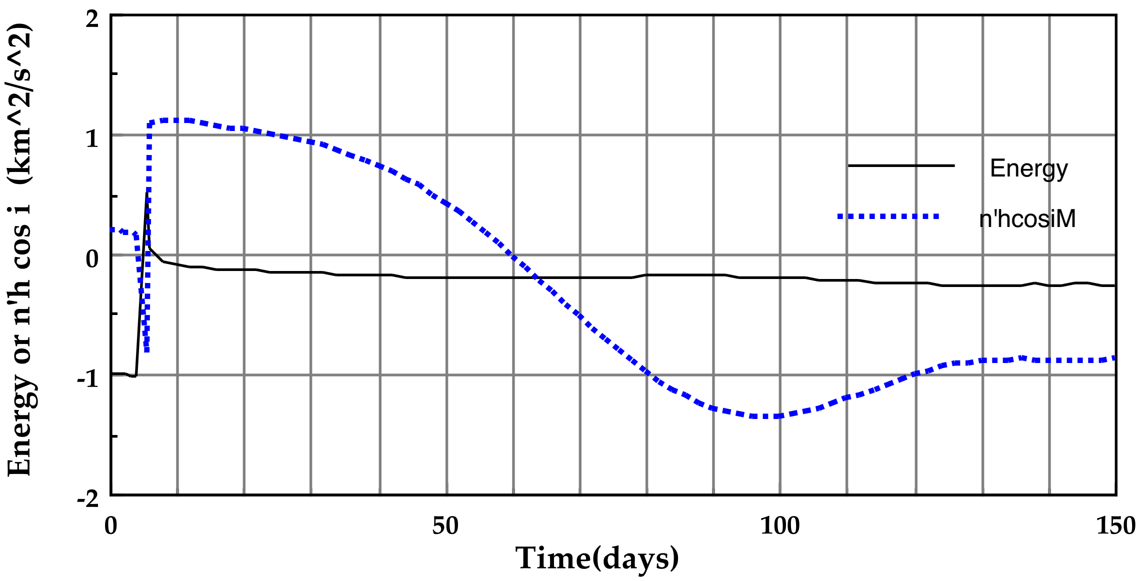

Fig. 6a Time History of Energy and Axial Angular Momentum

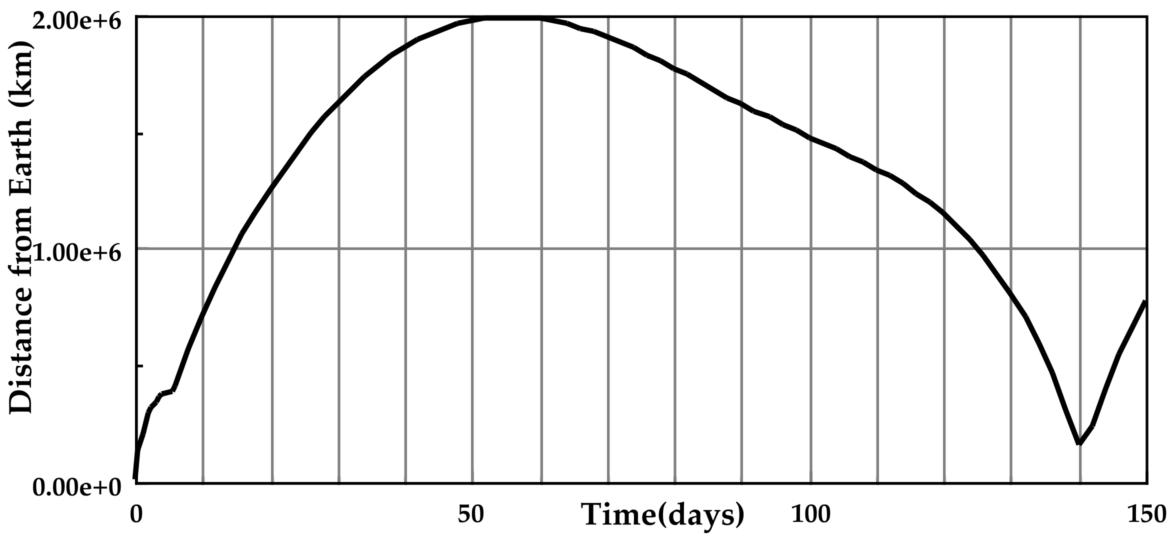

Fig. 6b Time History of Distance from Earth

Now. in the next sequence of figures, we examine the variations of the Jacobian "constants" in the two separate circular restricted three-body problems inherent in the execution of the triple lunar swingby. Fig. 7 shows the "constant"*C*s, the Jacobian constant in the Sun/Earth-moon system, while Fig. 8 shows the quantity *C*, for the Earth-moon system, as calculated from the numerical integrations.

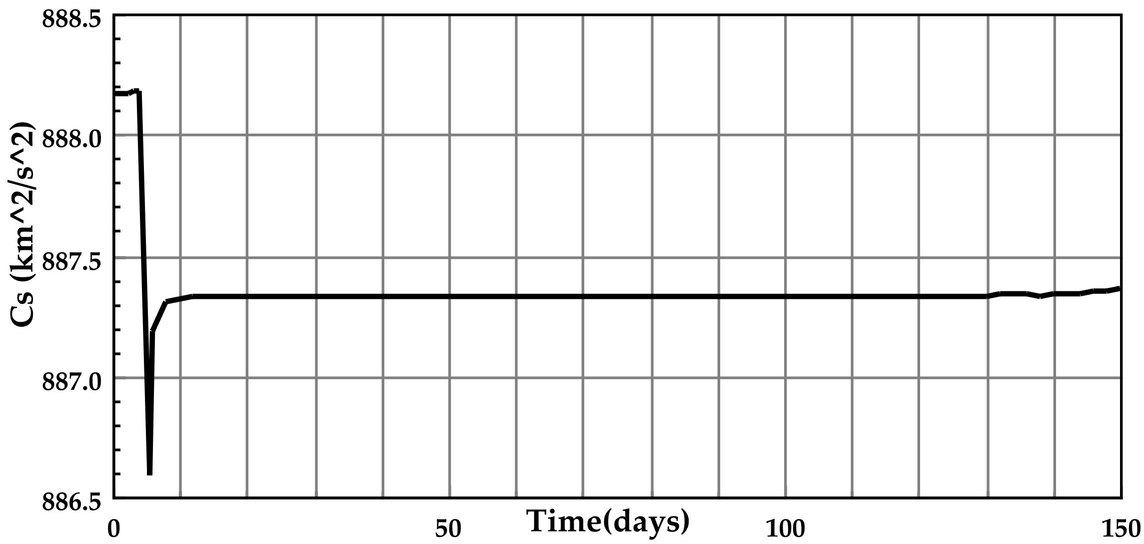

Fig. 7 Time History of Sun-Earth/moon Jacobian Constant for TLS Phase S_1-S_2

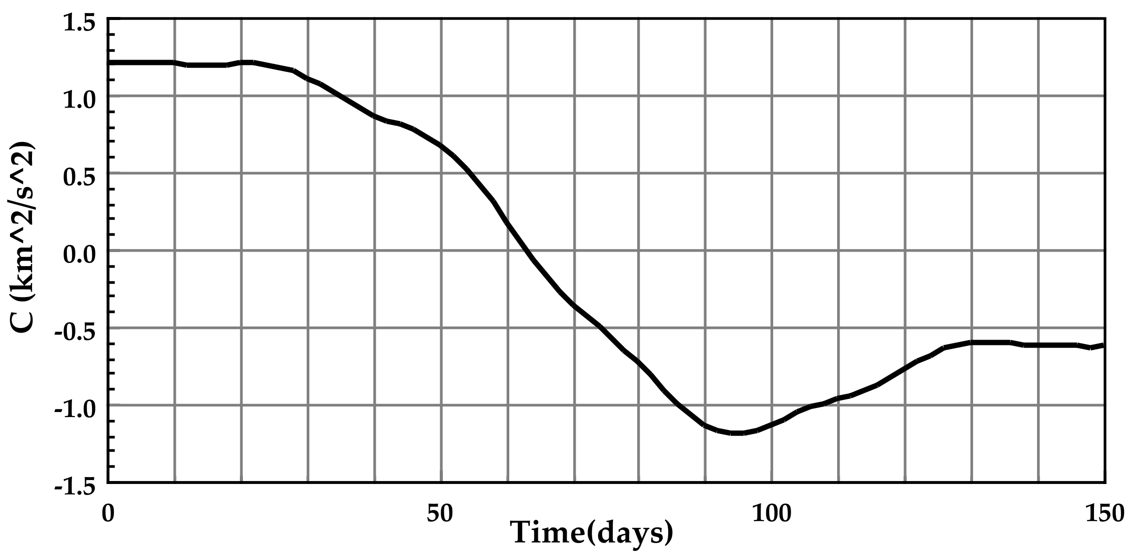

Fig. 8 Time History of Earth-moon Jacobian Constant for TLS Phase S_1-S_2

Notice, particularly in Fig. 8, how the Jacobian constant for the Earth-moon system is nearly constant for the first 20 days, then begins a substantial decay due to the Solar perturbations during passage through the 1st quadrant of Fig. 5, reversing the trend as the spacecraft passes into the 4th quadrant, and then settling down to a near-constant value as the spacecraft returns to the vicinity of the Earth's gravitational dominance.

Our knowledge of Cs permits us to estimate the required value of the Earth relative energy when the spacecraft returns to Earth after its excursion to the transition region, if we insist that the orbit have a certain minimum value of (negative) angular momentum in order to ensure the viability of the retrograde double lunar swingby S_2-S_3. We know, from the two-body analysis of the retrograde double lunar swingby, that the perigee radius of the (retrograde) Earth orbit after return from the transition region should be of the order of 35 to 40 earth radii (220,000 km to 260,000 km.) When the spacecraft returns to the vicinity of the moon's orbit, where its dynamics is substantially controlled by the Earth's gravitational field, one can estimate the amount of change in axial angular momentum required by the solar perturbations during the S_1-S_2 phase of the TLS. A simple scheme of iteration or the solution of a quadratic approximation will yield the energy of the return orbit.

The author has noticed, in this return to familiar analytic/geometric dynamics, that the two superimposed problems of three bodies, can be combined in the following simple approximation:

Jacobi's constant in the Earth-moon system, when the spacecraft is not too near the Sun, can be approximated by

$$C = n' \, h \cos i - \left[\frac{V^2}{2} - \frac{\mu}{r} \right] + \frac{\mu'}{a'} ,$$

and the dynamically equivalent quantity for the Sun/Earth-moon system, when the spacecraft is not too near the moon, is

$$C_s = n_s \, h \cos i - \left[\frac{V^2}{2} - \frac{\mu}{r} \right] + \frac{\mu_s}{a_s} .$$

Clearly, in this approximation,

$$C - C_s = \left[n' - n_s \right] h \cos i + \frac{\mu'}{a'} - \frac{\mu_s}{a_s} ,$$

and we can relate the change in one or the other of the Jacobian "constants" to the change in Earth-relative axial angular momentum. In the numerous situations where the effects of the two perturbations are spatially isolated, one can determine the amount of change in axial angular momentum, to be brought about by the solar perturbations (for example) that will result in a desired change in the Jacobian "constant" that is affected in this realm of the dynamical system.

More succinctly, when the spacecraft is dominated by the Sun/Earth-moon system, there will be very little change in Cs. When the spacecraft is dominated by the Earth-moon system, there will be very little change in C. This knowledge permits the analyst to approximate the change in one or the other of the Jacobian "constants" in terms of the change in axial angular momentum. The change in Earth-relative energy will come from the determination of the appropriate Jacobian "constant."

Comments on Chaos

The cover of this paper is symbolic only. It represents a hope of great discoveries in the connections between classical dynamics and the new Science of Chaos. The name is a controversial choice of word to be sure, but the concept is a major step in Physics, the continuous striving, by volitional beings, to describe the Universe. The cover diagram shows a completely deterministic trajectory in the circular restricted three-body problem whose path in the rotating coordinate frame is nearly identical to the curve of the boundary of the Mandelbrot set (M) near (-0.252696, 0.000238*i*) at magnification ≈ 40,000. The boundary curve of M has been superimposed on the three-body trajectory to represent a transition from the old, deterministic, *locally convergent* mindset of astrodynamicists to the possibility of a middle ground between classical and quantum mechanics where some, but not all, information can be effectively transformed forward in time and space.

Many of the participants of this conference are mavericks - those who march to a different drummer, or those who won't march at all. It is no accident that Dreamers tend to congregate rarely, and for important ideas. The Establishment has rarely supported Dreamers until their Dreams have been "Common Knowledge" for generations. The theory of Chaos, perhaps more than most new ideas, was born in an atmosphere of maverickism, by Dreamers.

It is here, in conferences like this, where the dreams of people like Mandelbrot, Feigenbaum, Lorenz, and May, are transformed into "Common Knowledge." It is the author's hope that these suggestions presented in this paper for simplifying the non-linearities of the restricted four-body problem will be useful in the future development of the theory of non-linear dynamics.

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