DERIVATION OF THE B-PLANE (BODY PLANE) AND ITS ASSOCIATED PARAMETERS

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I. INTRODUCTION

When sending a spacecraft to some extraterrestrial body, one navigation and mission design metric used is B-plane (Body Plane) targeting. The spacecraft's target body miss distance (with its associated covariance) and the target body's impact radius, can be readily mapped onto this B-plane. Things such as the performance of trajectory correction maneuvers (TCMs) and the effects of dynamic model errors (as well as perturbative effects of angular momentum desaturations {AMDs}) upon the trajectory, can be characterized and optimized in terms of B-plane parameters. It is herein that the advantage of it use is sought and found.

Simply stated, the B-plane can be defined as a plane which is normal to the incoming asymptote of the hyperbolic orbit and contains the target body's center of mass or, equivalently, normal to the velocity vector at "infinity" where infinity is defined to be far enough away from the vertex of the hyperbola, such that the trajectory essentially lies on the asymptote[2]. The B-vector is defined to be the vector from the origin (target body's center of mass) to that point where the asymptote (or v_∞ vector) intersects the B-plane.

II. CONSERVATION OF MECHANICAL ENERGY AND ANGULAR MOMENTUM (IN THE ABSENCE OF NON-CONSERVATIVE FORCES)

Assume the restricted two-body problem such that:

$$\ddot{\mathbf{r}} = -\frac{m}{r^3}\mathbf{r} \qquad (1)$$

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Taking the scalar product of (1) by $\dot{\mathbf{r}}$:

$$\dot{\mathbf{r}} \bullet \ddot{\mathbf{r}} + \dot{\mathbf{r}} \bullet \frac{m}{r^3}\mathbf{r} = 0$$

In general $\mathbf{a} \bullet \dot{\mathbf{a}} = a\dot{a}$, and letting $\mathbf{v} = \dot{\mathbf{r}}$ and $\dot{\mathbf{v}} = \ddot{\mathbf{r}}$:

$$\mathbf{v} \bullet \dot{\mathbf{v}} + \frac{m}{r^3}\mathbf{r} \bullet \dot{\mathbf{r}} = 0, \text{ thus}$$

$$v\dot{v} + \frac{m}{r^2}\dot{r} = 0$$

Taking note of the fact that $\frac{d}{dt}\left(\frac{v^2}{2}\right) = v\dot{v}$ and $\frac{d}{dt}\left(c - \frac{m}{r}\right) = \frac{m}{r^2}\dot{r}$

$$\frac{d}{dt}\left(\frac{v^2}{2} + c - \frac{m}{r}\right) = 0$$

The term on the left is the spacecraft's kinetic energy per unit mass, and the two terms on the right are the potential energy per unit mass, with c being an arbitrary constant that defines our potential energy datum (which we choose to be at a distance r = infinity making $c = \frac{m}{\infty} = 0$). Thus (taking the time rate of change of the previous expression to be zero) we are left with a constant (i.e. specific mechanical energy is conserved), being the familiar *vis-viva* equation:

$$e = \frac{v^2}{2} - \frac{m}{r} \qquad (2)$$

Now, if we take the vector product of (1) by $\mathbf{r}$:

$$\mathbf{r} \times \ddot{\mathbf{r}} + \mathbf{r} \times \frac{m}{r^3}\mathbf{r} = 0$$

Since $\mathbf{r} \times \mathbf{r} = 0$, we are left with

$$\mathbf{r} \times \ddot{\mathbf{r}} = 0$$

Taking note of the fact that $\frac{d}{dt}(\mathbf{r} \times \dot{\mathbf{r}}) = \dot{\mathbf{r}} \times \dot{\mathbf{r}} + \mathbf{r} \times \ddot{\mathbf{r}}$, the previous equation becomes

$$\frac{d}{dt}(\mathbf{r}\times\dot{\mathbf{r}}) = 0 \text{ or } \frac{d}{dt}(\mathbf{r}\times\mathbf{v}) = 0$$

Thus, being that the time rate of change of the previous expression is equal to zero, we are left with a constant (i.e. specific angular momentum is conserved):

$$\mathbf{h} = \mathbf{r}\times\mathbf{v} \quad (3)$$

Now, if we take the vector product of (1) into **h**:

$$\ddot{\mathbf{r}}\times\mathbf{h} + \frac{m}{r^3}(\mathbf{r}\times\mathbf{h}) = 0$$

Taking note of the fact that $\frac{d}{dt}(\dot{\mathbf{r}}\times\mathbf{h}) = \ddot{\mathbf{r}}\times\mathbf{h} + \dot{\mathbf{r}}\times\dot{\mathbf{h}}$, and recalling that **h =** const:

$$\frac{d}{dt}(\dot{\mathbf{r}}\times\mathbf{h}) + \frac{m}{r^3}(\mathbf{r}\times\mathbf{h}) = 0$$

However, $-\frac{m}{r^3}(\mathbf{r}\times\mathbf{h}) = \frac{m}{r^3}(\mathbf{h}\times\mathbf{r}) = \frac{m}{r^3}((\mathbf{r}\times\mathbf{v})\times\mathbf{r}) = \frac{m}{r^3}[\mathbf{v}(\mathbf{r}\bullet\mathbf{r}) - \mathbf{r}(\mathbf{r}\bullet\mathbf{v})]$. Since in general $\mathbf{a}\bullet\dot{\mathbf{a}} = a\dot{a}$, we are left with:

$$\frac{m}{r^3}(\mathbf{h}\times\mathbf{r}) = \frac{m}{r}\mathbf{v} - \frac{m}{r^2}\dot{r}\mathbf{r}$$

Taking note of the fact that $m\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right) = \frac{m}{r}\mathbf{v} - \frac{m}{r^2}\dot{r}\mathbf{r}$, then we can write

$$\frac{d}{dt}(\dot{\mathbf{r}}\times\mathbf{h}) = m\frac{d}{dt}\left(\frac{\mathbf{r}}{r}\right)$$

Integrating both sides of the previous expression, we are left with

$$\dot{\mathbf{r}}\times\mathbf{h} = m\left(\frac{\mathbf{r}}{r}\right) + \mathbf{P} \qquad (4)$$

Where **P** is a vector constant of integration which happens to point in the direction of the hyperbola vertex (i.e. periapse as defined in a perifocal coordinate frame) and has magnitude equal to $-v_\infty^2 ae$ (where ae is the hyperbolic semi-major axis and eccentricity, respectively). Now, if we take the vector product of (4) by **r**:

$$\mathbf{r}\bullet\dot{\mathbf{r}}\times\mathbf{h}=\mathbf{r}\bullet m\left(\frac{\mathbf{r}}{r}\right)+\mathbf{r}\bullet\mathbf{P}$$

Generally, $\mathbf{a}\bullet\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}\bullet\mathbf{c}$ and $\mathbf{a}\bullet\mathbf{a}=a^2$, therefore

$$h^2 = mr + rP\cos f$$

Where f is the angle between $\mathbf{P}$ and $\mathbf{r}$ (which is also known as the true anomaly). Solving for r, from the previous expression:

$$r=\frac{h^2/m}{1+\left(P/m\right)\cos f} \qquad (5)$$

We are left with an expression which the reader may recognize as being the polar form of a conic section.

III. DERIVATION OF THE B-PLANE AND ASSOCIATED PARAMETERS

Let us now define a unit vector $\hat{\mathbf{S}}$ which has its origin at the target body's center of mass and is parallel to the hyperbolic approach asymptote. We will be able to express $\hat{\mathbf{S}}$ in terms of a perifocal coordinate frame. First, we may state that:

$$\hat{\mathbf{S}}\bullet\mathbf{P}=-P\cos f_\infty$$

Where $\cos f_\infty = \lim_{r\to\infty}\cos f = \lim_{r\to\infty}\left[\frac{h^2-mr}{rP}\right]=-\frac{m}{P}$, and the negative sign takes care of the fact that f will be greater than 180 degrees at infinity. So, if the vector $\mathbf{Q}$ is defined as $\mathbf{Q}=\mathbf{h}\times\mathbf{P}$ (i.e. as in the perifocal coordinate frame), then we may express $\hat{\mathbf{S}}$ as follows:

$$\hat{\mathbf{S}}=-\left[\cos f_\infty\frac{\mathbf{P}}{P}+\sin f_\infty\frac{\mathbf{Q}}{Q}\right] \qquad (6)$$

Where $\sin f_\infty = -\sqrt{1-\frac{m^2}{P^2}}$.

Recall that the B-vector ($\mathbf{B}$) is defined to be the vector from the origin to that point where the asymptote (or v_∞ vector) intersects the B-plane. One can also think of this vector as being the radius of closest approach to the target body if the target body were massless. Based on the definition of **B**, it must satisfy the following relations:

$$\mathbf{B}\times\left(v_\infty\hat{\mathbf{S}}\right)=\mathbf{h} \qquad (7)$$

$$\mathbf{B}\bullet\left(v_\infty\hat{\mathbf{S}}\right)=0 \qquad (8)$$

Now, if we take the vector product of (7) by $\hat{\mathbf{S}}$:

$$\hat{\mathbf{S}}\times\left(\mathbf{B}\times\hat{\mathbf{S}}\right)=\frac{1}{v_\infty}\left(\hat{\mathbf{S}}\times\mathbf{h}\right)$$

or

$$\left(\hat{\mathbf{S}}\bullet\hat{\mathbf{S}}\right)\mathbf{B}-\left(\hat{\mathbf{S}}\bullet\mathbf{B}\right)\hat{\mathbf{S}}=\frac{1}{v_\infty}\left(\hat{\mathbf{S}}\times\mathbf{h}\right)$$

Since **B** and $\hat{\mathbf{S}}$ are mutually orthogonal (8) and the fact that $\hat{\mathbf{S}}$ is a unit vector, we have that:

$$\mathbf{B}=\frac{1}{v_\infty}\left(\hat{\mathbf{S}}\times\mathbf{h}\right) \qquad (9)$$

Using expression (6), **B** can also be written as:

$$\mathbf{B}=-\frac{h}{v_\infty}\left[\sin f_\infty\frac{\mathbf{P}}{P}-\cos f_\infty\frac{\mathbf{Q}}{Q}\right]$$

where the magnitude of **B** (since **B**, $\hat{\mathbf{S}}$, and **h** are mutually orthogonal) is simply

$$B=\frac{h}{v_\infty} \qquad (10)$$

However, if we take note of the fact that

$$h=rv\cos g=r_p v_p \qquad (11)$$

where γ is the spacecraft's flight path angle with respect to the target body and $r_p v_p$ are the spacecraft's periapse radius and velocity magnitudes respectively, then we can substitute (11) into (10) and obtain

$$B=\frac{r_p v_p}{v_\infty}$$

Yet, recognizing that

$$v_\infty^2 = v_p^2 - \frac{2m}{r_p}$$

We can now express B as

$$B = \frac{r_p\sqrt{v_\infty^2 + \frac{2m}{r_p}}}{v_\infty} = \frac{r_p\sqrt{v_\infty^2\left(1 + \frac{2m}{v_\infty^2 r_p}\right)}}{v_\infty} = r_p\sqrt{1 + \frac{2m}{v_\infty^2 r_p}} \qquad (12)$$

Solving for r_p from (12), we get a quadratic equation in the same:

$$r_p^2 + \frac{2m}{v_\infty^2} r_p - B^2 = 0$$

Which works out to become

$$r_p = -\frac{m}{v_\infty^2} + \sqrt{\left(\frac{m}{v_\infty^2}\right)^2 + B^2} \qquad (13)$$

As for the actual B-plane, let us define the following vectors:

> $\mathbf{T} \equiv$ a vector normal to $\hat{\mathbf{S}}$ lying along the reference plane, which typically is the target body's equatorial plane or parallel to either the ecliptic or Earth mean equator of J2000

Then let

$$\mathbf{R} = \hat{\mathbf{S}} \times \mathbf{T}$$

In this manner, $\mathbf{R}$, $\hat{\mathbf{S}}$, and $\mathbf{T}$ form the basis of an orthogonal coordinate system with $\hat{\mathbf{S}}$ being normal to the B-plane and the other two lying within it. For navigation, we usually think of the miss parameter (**B**) as being broken down into two components:

$$B1 = \mathbf{B} \bullet \mathbf{R} \qquad (14)$$

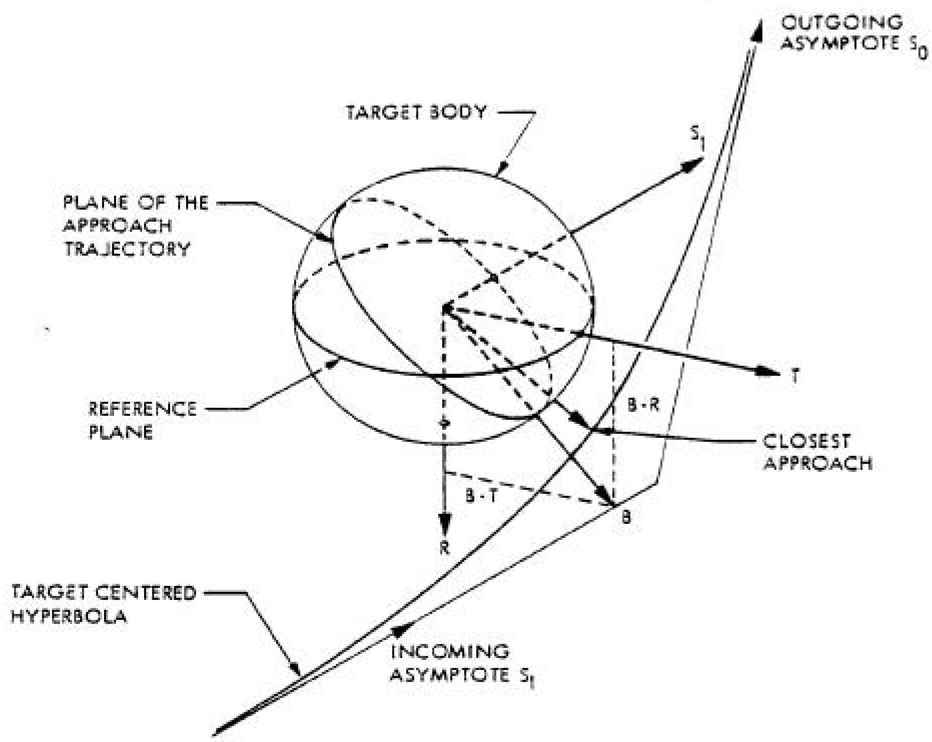
$$B2 = \mathbf{B} \bullet \mathbf{T} \qquad (15)$$

From (9), we can rewrite these expressions as

$$B1 = \frac{1}{v_\infty}\left(\hat{\mathbf{S}}\times\mathbf{h}\right)\bullet\mathbf{R} = \frac{1}{v_\infty}\left(\mathbf{R}\times\hat{\mathbf{S}}\right)\bullet\mathbf{h} = \frac{1}{v_\infty}\left(\mathbf{T}\bullet\mathbf{h}\right)$$

$$B2 = \frac{1}{v_\infty}\left(\hat{\mathbf{S}}\times\mathbf{h}\right)\bullet\mathbf{T} = \frac{1}{v_\infty}\left(\mathbf{T}\times\hat{\mathbf{S}}\right)\bullet\mathbf{h} = -\frac{1}{v_\infty}\left(\mathbf{R}\bullet\mathbf{h}\right)$$

The following figure shows the geometry of the B-plane and its associated parameters[3]



Associated with the B-plane, is a Linearized Time Of Flight (LTOF) and Time of Closest Approach (TCA). The LTOF and TCA are exactly the same if the target body were massless (i.e. LTOF is the TCA for the rectilinear approach trajectory along the incoming asymptote). The uncertainty in LTOF is one of the B-plane targeting performance metrics. In short, the spacecraft will have a 3-dimensional covariance mapped to the B-plane, with a component in the $\hat{\mathbf{S}}$ direction. This component is what effectively can be considered as the uncertainty in LTOF (σ_{LTOF}). The hyperbolic TCA can be computed as:

$$TCA = T_{reference\ epoch} + \frac{f - e\sin f}{\sqrt{\frac{-a^3}{m}}}$$

Where all the elements have been previously defined

IV. DEFINITION OF B-PLANE STATISTICS

The navigation error ellipse, upon the B-plane, defines the formal uncertainty about the B-plane aimpoint. A Gaussian distribution is used when formulating these statistics. The statistics associated with both components of **B** (i.e. B1 and B2 as previously defined) are assumed to follow a binormal (two-dimensional) distribution with the following probability density function:

$$f_{RT} = \frac{1}{2\pi\sigma_R\sigma_T\sqrt{1-\rho_{RT}^2}}\exp\left\{-\frac{\left[\left(\frac{B1}{\sigma_R}\right)^2 - 2\rho_{RT}\left(\frac{B1}{\sigma_R}\right)\left(\frac{B2}{\sigma_T}\right)+\left(\frac{B2}{\sigma_T}\right)^2\right]}{2\left(1-\rho_{RT}^2\right)}\right\}$$

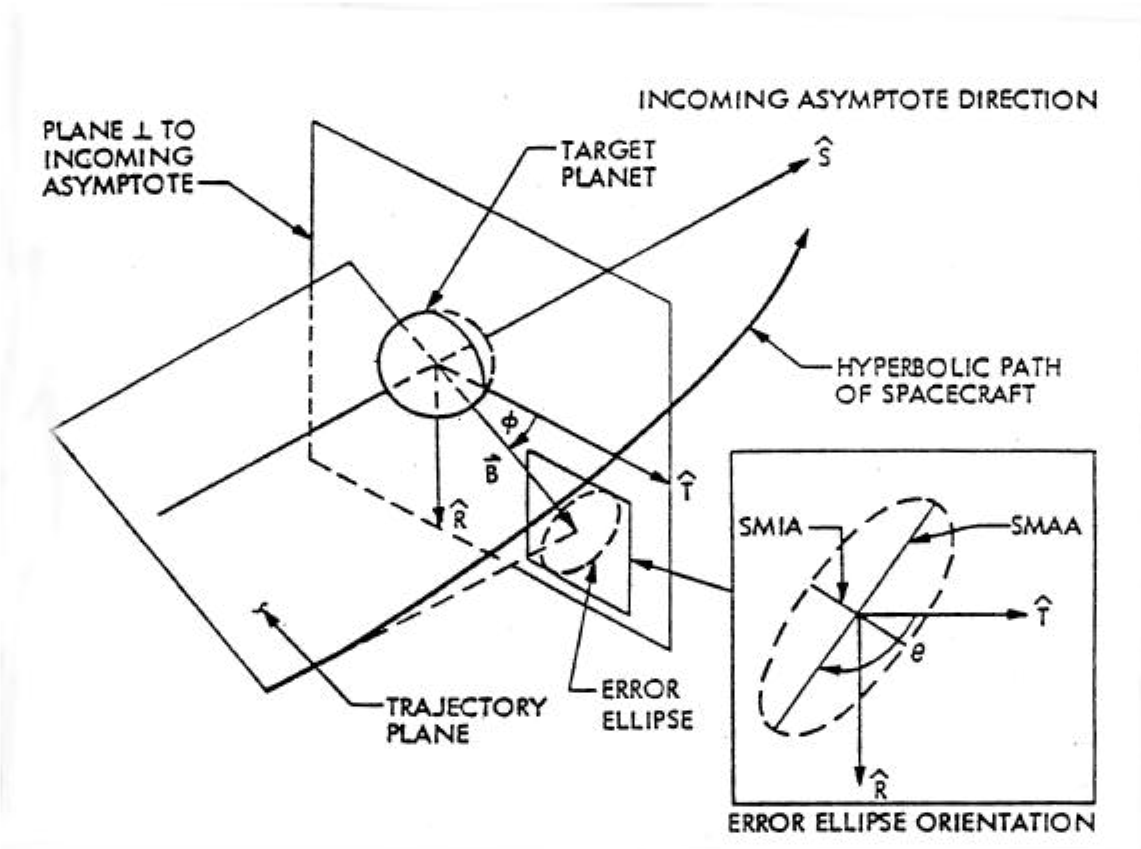
Where σ_R, σ_T, and ρ_{RT} are the standard deviations for each component and their associated correlation coefficient, respectively. Error ellipses are used because contours of constant values of f_{RT} happen to be ellipses, for this type of distribution. The size and orientation of a given probability ellipse is defined by σ_R, σ_T, and ρ_{RT}. The standard deviations are typically 1-σ values, but for targeting purposes are multiplied by 3 to get the error ellipses in terms of 3-σ values.

The correlation coefficient is not typically given because the components of the ellipse can be transformed (rotated) to a frame in which the correlation coefficient goes to zero. In this frame, the orientation of the ellipse is given as an angle between the original **R** and **T** axes (θ: measured from **T** positive clockwise)[2]. This coordinate frame transformation is done by diagonalizing the covariance matrix. Basically, for any square matrix **A** that has a basis of eigenvectors, then

$$\mathbf{D} = \mathbf{X}^{-1}\,\mathbf{A}\,\mathbf{X}$$

is a diagonal matrix with the eigenvalues of **A** as the entries on the main diagonal. Here, **X** is the matrix with the associated eigenvectors as column vectors[5]. For navigation purposes, if **A** is the original covariance matrix mapped to the B-plane aimpoint (the upper 2 x 2 portion because the 3rd component relates to σ_{LTOF} as previously discussed), then **D** is the matrix with a rotated covariance such that the correlation coefficient is zero. The square root of the variances of this rotated covariance matrix (i.e. the diagonal elements of **D**) will be the semi-major (SMAA) and semi-minor (SMIA) axes of the transformed probability error ellipse.

s_R and s_T can be expressed in terms of the SMAA and SMIA of the probability error ellipse as shown in the figure below.



The standard deviations, s_R and s_T, are related to SMAA and SMIA as follows

$$s_R = \sqrt{(SMAA \sin \boldsymbol{q})^2 + (SMIA \cos \boldsymbol{q})^2}$$

$$s_T = \sqrt{(SMAA \cos \boldsymbol{q})^2 + (SMIA \sin \boldsymbol{q})^2}$$

For this type of distribution, the probability of being inside a Nσ error ellipse (where N is the number of sigma desired {1,2,3,…}) is given by the following equation:

$$p(N\boldsymbol{s}) = 1 - e^{-\frac{N^2}{2}}$$

Several values resulting from this probability function are given in the table below. Note that they are ***not*** the same as one-dimensional Gaussian values.

N	***P(N σ)***
1	39.35 %
2	86.47 %
3	98.89 %
4	99.97 %

The closest approach radius (radius of periapse r_p) uncertainty can be found from

$$\sigma_{r_p} = \frac{B}{r_p + \frac{\mu}{v_\infty^2}} \sigma_B$$

Where

$$\sigma_B^2 = \frac{1}{B^2}\left(\mathbf{B}^T \mathbf{C}\,\mathbf{B}\right)$$

$$\mathbf{C} = \begin{bmatrix} \sigma_R^2 & \rho_{RT}\sigma_R\sigma_T \\ \rho_{RT}\sigma_R\sigma_T & \sigma_T^2 \end{bmatrix}$$

V. CONCLUDING REMARKS

To map the spacecraft state and associated covariance to the B-plane, the analyst must use a state transition matrix and map to the desired time, and perform a coordinate frame rotation (typically from inertial to S-T-R). Then one must formulate partial derivatives of the B-plane parameters of interest with respect to the spacecraft state (or independent variables of interest). This allows for the analyst to relate changes in the state (i.e. perturbations due to solar pressure, TCMs, AMDs, model errors, etc.) to changes upon the B-plane parameters. These serve as measures to the error sensitivity of a trajectory. The partials are not derived here, but are left to the reader to either find in the various references or to be self-derived.

VI. REFERENCES

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